EXERCISE – IV

HINTS & SOLUTIONS

Sol.1 $\int_0^{2\pi} e^x \left(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right).dx$

$= \frac{1}{\sqrt{2}} \int_0^{2\pi} e^x \, (\underset{f(x)}{\underset{\downarrow}{\cos x}} - \underset{f'(x)}{\underset{\downarrow}{\sin x}}).dx$

Sol.2 $I = \int_0^{\pi/4} \frac{\cos x - \sin x}{10 + \sin 2x}.dx$

$= \int_0^{\pi/4} \frac{\cos x - \sin x}{9 + (\cos x + \sin x)^2}.dx$

$\sin x + \cos x = t$

$(\cos x - \sin x).dx = dt$

Sol.3 Use king and add.

$2I = \int_0^{\pi} (a\pi + 2b).\frac{\sec x.\tan x}{3 + \sec^2 x}.dx$

Let $\sec x = t$

$(\sec x.\tan x).dx = dt$

Sol.4 Use king and add.

$2I = (2\pi + 3) \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x}.dx$

Let con $x = t$

$(\sin x).dx = -dt$

Sol.5 Let $I = \int_0^{p+q\pi} |\cos x|\, dx$

$= \int_0^{q\pi} |\cos x|\, dx + \int_0^{p+q\pi} |\cos x|\, dx$

$= q\int_0^{\pi} |\cos x|\, dx + \int_0^{p} |\cos x|\, dx$

$\{\because$ period of $|\cos x|$ is $\pi\}$

$= q\left\{\int_0^{\pi/2} |\cos x|\, dx + \int_{\pi/2}^{\pi} |\cos x|\, dx\right\}$

$= \int_0^{p} \cos x\, dx$

$= q\left\{\int_0^{\pi/2} \cos x\, dx - \int_{\pi/2}^{\pi} \cos x\, dx\right\} + \int_0^{p} \cos x\, dx$

$= q\,\{(\sin x)_0^{\pi/2} - (\sin x)_{\pi/2}^{\pi}\} + (\sin x)_0^{p}$

$= q\{(1 - 0) - (0 - 1)\} + \sin p - \sin 0 = 2q + \sin p$

Sol.6 I., $= \int_0^1 \frac{\sin^{-1}\sqrt{1-x}}{x^2 = x - 1}$

Let $\sin t = w$

$I = \int_0^1 \frac{2w}{w^4 - w^2 + 1}.dw$

Let $w^2 = z$

$I = \int_0^1 \frac{dz}{z^2 - z + 1}$

Sol.7 $I = \int_0^{\pi/2} \tan^{-1}\left[\frac{|\sin x/2 + \cos x/2| + |\sin x/2 - \cos x/2|}{|\sin x/2 + \cos x/2| - |\sin x/2 - \cos x/2|}\right].dx$

$= \int_0^{\pi/4} \tan^{-1}(\cot x/2)dx + \int_{\pi/4}^{\pi/2} \tan^{-1}(\cot x/2).dx$

Sol.8 Let $x^2 = t$

$I = \frac{1}{2} \int_{\frac{3a^2+b^2}{2}}^{\frac{a^2+b^2}{2}} \frac{dt}{(t-a^2)(b^2-t)}$

$= \frac{1}{2(b^2-a^2)} \int_{\frac{3a^2+b^2}{2}}^{\frac{a^2+b^2}{2}} \left\{\frac{1}{(t-a^2)} + \frac{1}{(b^2-t)}\right\}.dt$

Sol.9 $x^2 + 2x = k + \int_0^{-k} -(t+k)\,dt + \int_{-k}^{0} (t+k)\,dt$

$= 2k + k^2 + 1/2$

$\Rightarrow\ 2x^2 + 4x - (k + 2k^2 + 1) = 0$

$D = 16 + 4(2k^2 + 4k + 1).2$

$\Downarrow$

$D' < 0$

so $D > 0 \Rightarrow$ Real & Distinct roots $\forall\ x \in R$.

Sol.10 $\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{2a-x}{a}}\right) = t \Rightarrow x = 2a - 4a\sin^2 t$

$$I = \int_{\pi/4}^{0}(2a - 4a\sin^2 t).t\,(-8a\sin t.\cos t)\,dt$$

$$= a^2\int_0^{\pi/4}(16\sin t.\cos t - 32\sin^3 t.\cos t).t\,.\,dt$$

using king & add & then $(\pi/4 - t) = w$

$$2I = 16a^2\int_{\pi/4}^{0}(\sin w.\cos w - 2\sin^3 w.\cos w).\left(\frac{\pi}{4}\right)(-dw)$$

$$I = 2a^2\int_0^{\pi/4}(\sin w.\cos w - 2\sin^3 w.\cos w).dw$$

Sol.11 $u = \int_0^{\pi/4}\frac{1}{(1+\tan x)^2}.\frac{\sec^2 x}{(\sec^2 x)}.dx$

$$u = \int_0^1\frac{dt}{(1+t^2)(1+t)^2} = \frac{1}{2}\int_0^1\left(\frac{1}{1+t^2} - \frac{1}{(1+t)^2}\right).dt \quad ...(i)$$

$$v = \int_0^{\pi/4}(1+\tan x)^2 = \int_0^{\pi/4}(2 + \sec^2 x + 2\tan x)\,dx$$

Sol.12 $I = \int_0^{2\pi}\underset{\text{I}}{\underset{\downarrow}{x^2}}\left(\underset{\text{II}}{\underset{\downarrow}{\frac{\sin x}{9-\cos^2 x}}}\right).dx$ (using by parts)

IInd let $\cos x = t$

Sol.13 $\int_0^{\pi/4}\frac{x^2\,((1+\sin 2x) - (1 + (\cos 2x))}{(1+\sin 2x).\cos^2 x}.dx$

$$= \int_0^{\pi/4}x^2(\sec^2 x - \sec^2(\pi/4 - x).dx$$

Using by parts take x^2 as first function

$$= x^2(\tan x + \tan(\pi/4 - x))\Big|_0^{\pi/4}$$

$$-2\int_0^{\pi/4}x(\tan x + \tan(\pi/4 - x))\,dx$$

$$= \frac{\pi^2}{16} - 2\int_0^{\pi/4}x\,dx + 2\int_0^{\pi/4}x\tan x.\tan\left(\frac{\pi}{4} - x\right)dx$$

$$\left\{\because \frac{\pi}{4} = x + \left(\frac{\pi}{4} - x\right)\right\}$$

$$= \frac{\pi^2}{16} - \frac{\pi^2}{16} + 2\int_0^{\pi/4}x\tan x.\tan(\pi/4 - x)\,dx$$

use king & add

$$I = \frac{\pi}{4}\int_0^{\pi/4}\frac{\tan x(1-\tan x)}{(1+\tan x)}.dx$$

$$= \frac{\pi}{4}\int_0^{\pi/4}\left(-\tan x + 2 - \frac{2}{1+\tan x}\right).dx$$

$$= \frac{\pi}{4}(-\ell n\sec x) - \frac{\pi}{4}(x)_0^{\pi/4} - \frac{\pi}{2}\int_0^{\pi/4}\frac{dx}{1+\tan x}$$

$$= \frac{\pi^2}{16} - \frac{\pi}{4}\ell n\,2$$

Sol.14 Using by parts

$$\int_0^x\left(1\int_0^u f(t)\,dt\right)du = \left[\int_0^u f(t)\,dt\cdot u\right]_0^x$$

$$-\int_0^x\{f(u).u\}\,du$$

$$= x\int_0^x f(t)\,dt - \int_0^x uf(u)\,du$$

$$= x\int_0^x f(u)\,du - \int_0^x uf(u)\,du$$

$$= \int_0^x (x-u)\,f(u)\,du$$

Sol.15 $\frac{1}{4\sin x}\ \frac{d}{dx}\left(\frac{1}{5+4\cos x}\right) = \frac{1}{(5+4\cos x)^2}$

so $I = \int \underbrace{\frac{1}{4\sin x}}_{\mathrm{I}} . \underbrace{\frac{d}{dx}\left(\frac{1}{5+4\cos x}\right)}_{\mathrm{II}} .dx$

(By parts).

Sol.16 $I = \int_0^1 \ell n\left(\sqrt{1-x}+\sqrt{1+x}\right).dx \times \frac{2}{2}$

$I = \frac{1}{2}\int_0^1 \ell n\left(2+2\sqrt{1-x^2}\right).dx$

$= \frac{1}{2}\int_0^1 \ell n\, 2\, dx + \int_0^1 \ell n\left(1+\sqrt{1-x^2}\right).dx$

using by parts & take
1 as IInd function

so $I_1 = \int_0^1 1.\ell n\,(1+\sqrt{1-x^2})\,dx$

$= x\,\ell n\,(1+\sqrt{1-x^2})\Big|_0^1 - \int_0^1 \frac{(-x^2)}{\sqrt{1-x^2}\left(1+\sqrt{1-x^2}\right)}.dx$

Let x = sin θ & so on

Sol.17 put $\sqrt{\sqrt{x}-1} = t \Rightarrow \sqrt{x}-1 = t^2 \Rightarrow x = (t^2+1)^2$

$dx = 2(t^2+1)\,2t\,dt$

$I = \int_0^{\sqrt{3}} 4t(t^2+1).\tan^{-1} t\ dt$

$= \int_0^{\sqrt{3}} (4t^3+4t)\tan^{-1} t\ dt$ use by parts

$= \frac{16\pi}{3} - 2\sqrt{3}$

Sol.18 $I = \lim\limits_{n\to\infty} n^2 \int_{-1/n}^{1/n} 2006\underbrace{\sin x.|x|}_{\text{odd}}\,dx + \int_{-1/n}^{1/n} 2007\underbrace{\cos x.|x|}_{\text{even}}\,dx$

so $I = \lim\limits_{n\to\infty} 2\int_0^{1/n} 2007\cos x.x.dx$

$\lim\limits_{n\to\infty} 4014\left(\frac{\sin 1/n}{1/n} + \frac{\cos 1/n - 1}{1/n^2}\right) = 2007$

Sol.19 $\int_0^\infty f\left(\frac{a}{x}+\frac{x}{a}\right).\frac{\ell n\, x}{x}\,dx$ Let $a/x = t$

$= \int_0^\infty f\left(t+\frac{1}{t}\right).\left(\frac{\ell n\, a - \ell n\, t}{t}\right).dt$

$= \underbrace{\ell n\, a\int_0^\infty f\left(t+\frac{1}{t}\right).\frac{dt}{t}}_{t = a/x} - \underbrace{\int_0^\infty f\left(t+\frac{1}{t}\right)\frac{\ell n\, t}{t}\,dt}_{t = a/x\ \&\ \Rightarrow 0}$

$= \ell n\, a\int_0^\infty f\left(\frac{a}{x}+\frac{x}{a}\right).\frac{dx}{x}$.

Sol.20 $I = \int_{-1}^{1}\left(\frac{2x^{332}+x^{998}}{\underbrace{1+x^{666}}_{\text{even}}} + \frac{4x^{1668}.\sin x^{691}}{\underbrace{1+x^{666}}_{\text{odd}}}\right).dx$

so $I = 2\int_0^1 \frac{x^{332}}{1+x^{666}}\,dx + \int_0^1 \frac{x^{332}(1+x^{666})}{(1+x^{666})}\,dx$

$x^{333} = t$ & so on.

Sol.21 $\int_0^\infty \frac{dx}{a^2+(x-1/x)^2}$

$= \frac{1}{2}\int_0^\infty \frac{(1+1/x^2)+(1-1/x^2)}{a^2+(x-1/x)^2}\,dx$

$= \frac{1}{2}\left\{\underbrace{\int_0^\infty \frac{(1+1/x^2).dx}{(a^2)+(x-1/x)^2}}_{\text{let } x-1/x = t} + \underbrace{\int_0^\infty \frac{(1-1/x^2).dx}{(a^2-4)+(x+1/x)^2}}_{\text{let } x+1/x = t}\right\}$

Sol.22 $\int_0^1 e^{\ell n.\tan^{-1} x}\sin^{-1}(\cos x)\,dx$

$= \int_0^1 \tan^{-1} x\,\sin^{-1}(\sin(\frac{\pi}{2}-x))\,dx$

$= \int_0^1 \frac{\pi}{2}.\tan^{-1} x - \underbrace{\int_0^1 x\tan^{-1} x\,dx}_{\text{use by parts}}$

Sol.23 $\frac{d}{dx}(f(x)) = \frac{\cos x}{f(x)}$

$\int f(x).df(x) = \int \cos x\, dx$

$\frac{f^2(x)}{2} = \sin x + c$

$f(x) = \sqrt{2\sin x + c}$ so f(x) is periodic

Sol.24 $f(x) = \int_{-1}^{1} \frac{\sin x\, dt}{\sin^2 x + (t - \cos x)^2}$

$= \frac{\sin x}{\sin x} \tan^{-1}\left(\frac{t - \cos x}{\sin x}\right)_{-1}^{1}$

$= \tan^{-1}(\tan x/2) + \tan^{-1}(\cot x/2)$

Case–1 : $0 < x < \pi$

$f(x) = \pi/2$

Case–2 : $\pi < x < 2\pi$

$f(x) = -\pi/2$

so range $= \left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}$

Sol.25 $f'(x) = 2x \sin 1/x - \cos 1/x$

integrating by parts

$f(x) = \sin 1/x . x^2 - \int \cos\frac{1}{x}\left(-\frac{1}{x^2}\right).x^2\, dx$

$- \int \cos\frac{1}{x}\, dx + c$

$f(x) = x^2 \sin\frac{1}{x} + c\ ;\ c = 0$

$f(x) = x^2 \sin\frac{1}{x}$

continious & differentiable at x = 0.

Sol.26 $g(x) = \int_{-2}^{x} f(t)\, dt$

Taking different integrals

$$g(x) = \begin{cases} -(x+2)\ ;\ -2 \le x \le 0 \\ -2 + x - \frac{x^2}{2}\ ;\ 0 < x < 1 \\ \frac{x^2}{2} - x - 1\ ;\ 1 \le x \le 2 \end{cases}$$

Not differentiable at x = 0.

Sol.27 **(a)** $0 < x^3 < x^2$

$-2x^2 < -(x^2 + x^3) < -x^2$

$4 - 2x^2 < 4 - x^2 - x^3 < 4 - x^2$

$\Rightarrow \int_0^1 \frac{1}{\sqrt{4-x^2}} < \int_{-1}^{1} \frac{1}{\sqrt{4-x^2-x^3}}\, dx < \int_0^1 \frac{1}{\sqrt{4-2x^2}}\, dx$

$\Rightarrow \sin^{-1}\left(\frac{x}{2}\right)\Big|_0^1 < \int_0^1 \frac{1}{\sqrt{4-x^2-x^3}}\, dx < \frac{1}{\sqrt{2}} \sin^{-1}\frac{x}{\sqrt{2}}\Big|_0^1$

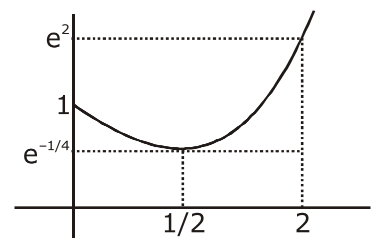
$\Rightarrow \frac{\pi}{6} < \int_0^1 \frac{1}{\sqrt{4-x^2-x^3}}\, dx < \frac{\pi}{4\sqrt{2}}$

(b) $f(x) = e^{x^2} - x$

$f'(x) = e^{x^2} - x\ (2x - 1) = 0 \Rightarrow x = \frac{1}{2}$

$e^{-1/4}(2, 0) < I < e^2(2, 0)$

$2e^{-1/4} < I < 2e^2$



(c) $-1 < \cos x < 1$

$-3 < 3\cos x < 3$

$7 < 10 + 3\cos x < 13$

$\frac{1}{13} < \frac{1}{10 + 3\cos x} < \frac{1}{7}$

$\frac{1}{13}\int_0^{2\pi} dx < I < \frac{1}{7}\int_0^{2\pi} dx$

$\frac{2\pi}{13} < I < \frac{2\pi}{7}$

(d) $I = \int_0^2 \frac{dx}{2 + x^2} = \frac{1}{\sqrt{2}} \tan^{-1}\frac{x}{\sqrt{2}} \int_0^2$

$= \frac{1}{\sqrt{2}} \tan^{-1}\sqrt{2}$

$\cong \frac{\pi}{4\sqrt{2}} \cong 0.555$

always lie in the given interval.

Sol.28 $y = \frac{1}{a}\int_0^1 f(t)\sin a(x-t)\,dt$

$y = \frac{1}{a}\int_0^x f(t)\,[\sin ax\cos at - \sin at\cos ax]\,dt$

$= \frac{\sin ax}{a}\int_0^x f(t).\cos at\,dt - \cos ax\int_0^x f(t)\sin at\,dt$

$\Rightarrow \frac{dy}{dx} = \frac{1}{a}\left\{a\cos ax\int_0^x f(t)\cos at\,dt + \sin ax(f(x)\cos ax\right.$

$\left. + a\sin ax\int_0^x f(t)\sin at\,dt - \cos ax\,(f(x)\sin ax)\right\}$

$= \cos ax\int_0^x f(t)\cos at\,dt + \sin ax\int_0^x f(t)\sin at\,dt$

$\Rightarrow \frac{d^2y}{dx^2} = -a\sin ax\int_0^x f(t)\cos at\,dt$

$+ a\cos ax\,(f(x) + \cos ax)$

$+ a\cos ax.\int_0^x f(t)\sin t\,dt + a\sin ax$

$[f(x)\sin ax] = -a^2y + y$

$\Rightarrow \frac{d^2y}{dx^2} + a^2y = f(x)$ HP.

Sol.29 $y = \int_1^x 1.\ell n\,t\,dt$

$= x^{x\ln x - x + 1}$

$\frac{dy}{dx} = y\left(\frac{1}{x}(x\,\ell n\,x - x + 1) + \ell n\,(\ell n\,x)\right)$

at $x = e$; $\frac{dy}{dx} = 1 + e$

Sol.30 $f(x) = x + x\int_0^1 y^2 f(y)\,dy + x^2\int_0^1 yf(y)\,dy$

Let $A = \int_0^1 y^2 f(y)\,dy$ & $B = \int_0^1 yf(y)\,dy$

Now $f(x) = x + Ax + Bx^2$

$f(y) = y + Ay + By^2$

$A = \int_0^1 y^2(y + Ay + By^2)\,dy$

$A = \int_0^1 (y^3 + Ay^2 + By^4)\,dy$

$A = \frac{1}{3} + \frac{A}{3} + \frac{B}{5} \Rightarrow \frac{3A}{4} - \frac{B}{5} = \frac{1}{4}$ ....(1)

$B = \int_0^1 y.(y + Ay + By^2)dy$

$\frac{A}{3} - \frac{3B}{4} = -\frac{1}{3}$ ....(2) Solve (1) & (2)

$A = \frac{61}{119}$ & $B = \frac{80}{119}$

Put the value of A & B in f(x)

Sol.31 **(a)** $L = \lim_{x\to\infty}\frac{f'(x)}{g'(x)} = \lim_{x\to\infty}\frac{e^{2x}x^c\left[\frac{c}{x}+2\right]}{e^{2x}(3x^2+1)^{1/2}}$

for the existance of limit $c - 1 = 0$

& $L = \frac{2}{\sqrt{3}}$ $\Rightarrow c = 1$

(b) $\lim_{x\to 0}\frac{\int_0^x \frac{t^2\,dt}{\sqrt{a+t}}}{bx - \sin x}$ ($\div$ form)

using L' Hospital

$L = \lim_{x\to 0}\frac{x^2}{(b-\cos x)\sqrt{a+x}}$

for the existance of limit : b = 1

& $L = \lim_{x\to 0}\frac{x^2}{(1-\cos x)}\sqrt{a+x} = 1 \Rightarrow a = 4$

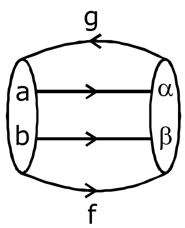
Sol.32 $\lim\limits_{x\to\infty} \dfrac{d}{dx} \displaystyle\int_{2\sin 1/x}^{3\sqrt{x}} \dfrac{3t^4+1}{(t-3)(t^2+3)}\,dt$

Using Libnitz theorem

$$L = \frac{3^5 x^2+1}{(3\sqrt{x}-3)(9x+3)}\cdot\frac{3}{2}\cdot\frac{1}{\sqrt{x}}$$

divide & multiply by x^2 & $L = 13.5$

Sol.33



$\alpha = f(a) \;\Rightarrow\; a = g(\alpha)$

$\beta = f(b) \;\Rightarrow\; b = g(\beta)$

Let $y = f(x)$ & $x = g(y)$ $\because$ g is the inverse of f

$dy = f'(x)\,dx$

$$\int_a^b f(x)\,dx + \int_\alpha^\beta g(y)\,dy = \int_a^b f(x)\,dx + \int_a^b x.f'(x)\,dx$$

$$= \int_a^b \{f(x) + x\,f'(x)\}\,dx = \left[xf(x)\right]_a^b$$

$= bf(b) - af(a) = b\beta - a\alpha$

Sol.34 **(a)** Taking log :

$$\ell n\, y = \lim_{n\to\infty} \frac{1}{n}\left\{\ell n\left(1+\frac{1}{n^2}\right) + \ell n\left(1+\frac{2^2}{n^2}\right) + ... + \ell n\left(1+\frac{h^2}{n^2}\right)\right\}$$

$$\ell n\, y = \int_0^1 \ell n\,(1+x^2)\,dx$$

$y = 2e^{1/2(\pi-4)}$

(b) $\lim\limits_{n\to\infty} \dfrac{1}{n}\displaystyle\sum_{r=1}^{3n} \dfrac{r}{n+r}$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{3n} \frac{r/n}{1+(r/n)}$$

$$I = \int_0^3 \frac{x}{1+x}\,dx = \int_0^3 \frac{x+1-1}{x+1}\,dx$$

$$= \int_0^3 dx - \int_0^3 \frac{dx}{x+1} = x\Big|_0^3 - \ln(1+n)\Big|_0^3$$

$I = 3 - \ln 4$

(d) (Let) $y = \lim\limits_{n\to\infty} \dfrac{A_n}{B_n} = \dfrac{\left(n.n+\dfrac{n(n+1)}{2}\right).\dfrac{1}{n}}{\left((n+1)(n+2)\ldots(n+n)\right)^{1/n}}$

$$= \frac{n+\dfrac{n+1}{2}}{n\left(\left(1+\dfrac{1}{n}\right)\left(1+\dfrac{2}{n}\right)....\left(1+\dfrac{n}{n}\right)\right)^{1/n}}$$

$$\ell n\, y = \lim_{n\to\infty}\left\{\ell n\left(\frac{3}{2}+\frac{1}{2n}\right) - \frac{1}{n}\ell n\left(\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)....\left(1+\frac{n}{n}\right)\right)\right\}$$

$$= \ell n\left(\frac{3}{2}\right) - \int_0^1 \ell n\,(1+x)\,dx$$

(c) Let $= \lim\limits_{n\to\infty} \ell n\left(\dfrac{1}{n}.\dfrac{2}{n}.\dfrac{3}{n}......\dfrac{n}{n}\right)^{1/n}$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\ell n\left(\frac{1}{n}\right) + \ell n\left(\frac{2}{n}\right) + ...\ell n\left(\frac{n}{n}\right)\right]$$

$= \displaystyle\int_0^1 \ell n\, x\,dx = -1$ $\quad A = e^{-1}$

Sol.35 $\sin x + \sin 3x + ... + \sin(2k-1)x$

$$= \frac{\sin\left(\dfrac{2x}{2}, k\right)}{\sin\left(\dfrac{2x}{2}\right)} \times \sin\left(\frac{x+(2k-1)x}{2}\right) = \frac{\sin^2 kx}{\sin x},\; k \in N$$

$$\int_0^{\pi/2} \frac{\sin^2 kx}{\sin x}.dx = \int_0^{\pi/2} \sin x\,dx$$

$$+ \int_0^{\pi/2} \sin 3x\,dx.... + \int_0^{\pi/2} \sin(2k+1)\,dx$$

$$= 1 + \frac{1}{3} + \frac{1}{5} + ... + \frac{1}{2k-1}$$

Sol.36 $x.\displaystyle\int_0^x y(t)\,dt = (x+1)\int_0^x t.y(t)\,dt$

Differentiating w.r.t. 'x'

$$\int_0^x y(t)\,dt + xy(x) = \int_0^x t\,y(t)\,dt + (x+1)\,x\,y(x)$$

$$\Rightarrow \int_0^x (1-t)\,y(t)\,dt = x^2 y(x)$$

again differentiating w.r.t. 'x'

$(1-x)y(x) = 2xy(x) + x^2y'(x)$

$y'(x) = y(x)\left(\frac{1-3x}{x^2}\right)$

Integrate & put y(1) = 1

$y(x) = \frac{e}{x^3}\, e^{-1/x}$

Sol.37 **(a)** $I_{m,n} = \int_0^1 \underbrace{x^m}_{\text{IInd}} \underbrace{(1-x)^n}_{\text{Ist}} dx$

$I_{m,n} = \left[\frac{x^{m+1}}{m+1}(1-x)^n\right]_0^1 + \frac{n}{m+1}$

$\int_0^1 x^{m+1}(1-x)^{n-1}\,dx$

& so on .... so $I_{m,n} = \frac{m!\,n!}{(m+n+1)!}$, m, n $\in$ N

(b) $I_{m,n} = \int_0^1 \underbrace{x^m}_{\text{IInd}} \underbrace{(\ell n\, x)^n}_{\text{Ist}} dx$

Sol.38 $f^2(x) = \int_0^x \{(f(t))^2 + (f'(t))^2\}\,dt + e^2$

differentiate both the side w.r.t. x

$2f(x)\,f'(x) = \{f(x)\}^2 + \{f'(x)\}^2 \cdot 1 + 0$

$\{f(x)\}^2 - 2f(x)\,f'(x) + \{f'(x)\}^2 = 0$

$\{f(x) - f'(x)\}^2 = 0 \;\Rightarrow\; f'(x) = f(x)$

$\Rightarrow \int \frac{f'(x)\,dx}{f(x)} = \int dx$

$\ln f(x) = x + c \;\Rightarrow\; f(x) = e^{x+c}$ ......(1)

$f(x) = e^x \,.\, e^c$

Now put x = 0 in given equation

$f^2(0) = c^2 \;\Rightarrow\; e^c = e^1 \;\Rightarrow\; c = 1$

put c = 1 in equation (1)

$f(x) = e^{x+1}$

Sol.39 $\int_1^x [t]\,f'(t)\,dt = \int_1^2 [t]\,f'(t)\,dt$

$+ \int_2^3 [t]\,f'(t)\,dt + \ldots\ldots + \int_n^{n+f} [t]\,f'(t)\,dt$

where n $\in$ N & $0 \le f < 1$

$= \int_1^2 f'(t)\,dt + 2\int_2^3 f'(t)\,dt + 3\int_3^4 f'(t)\,dt + \ldots + n\int_n^{n+f} f'(t)\,dt$

$= \left[f(t)\right]_1^2 + 2\left[f(t)\right]_2^3 + 3\left[t(t)\right]_3^4 + \ldots\ldots + n\left[f(t)\right]_n^{n+f}$

$= -f(1) - f(2) - f(3) \ldots.\, f(n) + nf(n+f)$

$= -[f(1) + f(2) + \ldots. + f(n)] + [x]\,f(x)$

$= [x]\,f(x) - \sum_{k=1}^{[x]} f(k)$

Sol.40 (FG)' = F'G + FG'

$=(\sqrt{4+x^2})\int_x^1 \sqrt{4+t^2}\,dt + (-\sqrt{4+x^2})\int_{-1}^x \sqrt{4+t^2}\,dt$

at x = 0

(FG)' = 0

Sol.41 $\because \int_0^x f'''(t)\,(x-t)^2\,dt$

$= [(x-t)^2 f''(t)]_0^x + 2\int_0^x (x-t)\,f''(t)\,dt$

$[-x^2 f''(0)] + 2\left[[(x-t)\,f'(t)]_0^x + \int_0^x f'(t)\,dt\right]$

$= -x^2 f''(0) - 2xf'(0) + 2f(x) - 2f(0)$

Replace this value is R.H.S.

$= xf'(0) + \frac{x^2 f''(0)}{2} - \frac{x^2}{2} f''(0) - xf'(0) + f(x) - f(0)$

= f(x) – f(0) = L.H.S. Hence proved

Sol.42 **(a)** $f'(0) = 4,\ g'(0) = -4$

$f'(x) = -g'(x)$

Integrating $\Rightarrow f(x) = -g(x) + c$

f(x) + g(x) = c at x = 0, c = 6

so f(x) + g(x) = 6

(b) Now $f'(x) = f(x) - g(x)$

$= 2f(x) - 6$

Integrating & using f(0) = 5

$\left.\begin{aligned} f(x) &= 3 + 2e^{2x} \\ \& \; g(x) &= 3 - 2e^{2x} \end{aligned}\right\}$